

Schwarz inequality and concurrence

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Abstract

We establish a relation between the Schwarz inequality and the generalized concurrence of an arbitrary, pure, bipartite or tripartite state. This relation places concurrence in a geometrical and functional-analytical setting.

1 Introduction

Quantum entanglement is one of the most interesting and debated properties of quantum mechanics. It has become an essential resource for the quantum communication created in recent years, with some potential applications such as quantum cryptography [1, 2] and quantum teleportation [3]. The idea of quantum entanglement goes back to the early days of quantum theory where it was initiated by Schrödinger, Einstein, Podolsky and Rosen [4, 5] and was later extended by Bell [6] in the form of Bell inequalities. Quantification of multipartite state entanglement [7, 8] is difficult and is a task that is directly linked to linear algebra, geometry and functional analysis. The definition of separability and entanglement of a multipartite state was introduced in [9] following the definition for bipartite states, given in 1989 by Werner [10]. One widely used measure of entanglement for a pair of qubits, is entanglement of formation [11]. A closely related measure is concurrence, that gives an analytic formula for the entanglement of formation [12]. In recent years, there have been several proposals to generalize this measure to general bipartite states, e.g., Uhlmann [13] has generalized the concept of concurrence by considering arbitrary conjugation, then Audenaert, Verstraete, and De Moor [14] generalized this formula in spirit of Uhlmann's work, by defining a concurrence vector for pure states. Another generalization of concurrence have been done by Rungta *et al.* [15] based on an idea of a superoperator called universal state inversion. And finally Gerjuoy, Albeverio and Fei, Akhtarshenas, and Bhaktavatsala and Ravishankar [16, 17, 18, 19] have given explicit expression in terms of the state amplitude coefficient of a pure bipartite state in any dimension.

In this paper, we put the concurrence in another perspective, namely we establish a relation between Schwarz' inequality and concurrence for bipartite states and then extend our connection to multipartite states. We show that, the generalized concurrence [17] and entanglement tensor [20] for a three-partite

state can be derived using the concept of Schwarz inequality. Generalization of this relation to a multipartite state with more than three subsystems can be tried out in the same way as for three-partite state but it gives only information about the set of separable state and can not quantify a general pure multipartite state completely.

2 Entanglement

In this section we will establish the notation for separable states and entangled states. Let us denote a general, pure, composite quantum system with m subsystems $\mathcal{Q} = \mathcal{Q}_m^p(N_1, N_2, \dots, N_m) = \mathcal{Q}_1 \mathcal{Q}_2 \cdots \mathcal{Q}_m$, consisting of a state

$$|\Psi\rangle = \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \cdots \sum_{i_m=1}^{N_m} \alpha_{i_1, i_2, \dots, i_m} |i_1, i_2, \dots, i_m\rangle \quad (1)$$

defined on a Hilbert space

$$\begin{aligned} \mathcal{H}_{\mathcal{Q}} &= \mathcal{H}_{\mathcal{Q}_1} \otimes \mathcal{H}_{\mathcal{Q}_2} \otimes \cdots \otimes \mathcal{H}_{\mathcal{Q}_m} \\ &= \mathbf{C}^{N_1} \otimes \mathbf{C}^{N_2} \otimes \cdots \otimes \mathbf{C}^{N_m}, \end{aligned} \quad (2)$$

where the dimension of the j th Hilbert space is given by $N_j = \dim(\mathcal{H}_{\mathcal{Q}_j})$. We are going to use this notation throughout this paper, i.e., we denote a pure pair of qubits by $\mathcal{Q}_2^p(2, 2)$. Next, let $\rho_{\mathcal{Q}}$ denote a density operator acting on $\mathcal{H}_{\mathcal{Q}}$. The density operator $\rho_{\mathcal{Q}}$ is said to be fully separable, which we will denote by $\rho_{\mathcal{Q}}^{sep}$, with respect to the Hilbert space decomposition, if it can be written as

$$\rho_{\mathcal{Q}}^{sep} = \sum_{k=1}^K p_k \bigotimes_{j=1}^m \rho_{\mathcal{Q}_j}^k, \quad \sum_{k=1}^K p_k = 1 \quad (3)$$

for some positive integer K , where p_k are positive real numbers and $\rho_{\mathcal{Q}_j}^k$ denote a density operator on Hilbert space $\mathcal{H}_{\mathcal{Q}_j}$. If $\rho_{\mathcal{Q}}^p$ represents a pure, fully separable state, then $K = 1$. If a state is not fully separable, then it is called an entangled state. A completely nonseparable quantum system is one that in any basis must be written

$$\rho_{\mathcal{Q}}^{nonsep} = \sum_{k=1}^K p_k \rho_{\mathcal{Q}}^k, \quad \sum_{k=1}^K p_k = 1, \quad (4)$$

where $\mathcal{Q} = \mathcal{Q}_1^p(N_1 N_2 \cdots N_m)$. The simplest such completely nonseparable, generic states (that, moreover, are maximally entangled) are Bell states and W-states.

3 The Schwarz inequality and concurrence

In this section, we will investigate the relation between concurrence, the Schwarz inequality, and the minors of determinant of bipartite states, which are directly related to the geometry of the Hilbert space and Segre variety [21].

Let us begin by reviewing the Schwarz inequality on an inner product space such as a Hilbert space, and then use this inequality to relate it to the geometry of concurrence.

Let $X_1 = (\xi_1, \xi_2)$ and $X_2 = (\eta_1, \eta_2)$ be two vectors defined on the complex Hilbert space $\mathcal{H} = \mathbf{C}^2$. Then X_1 and X_2 are parallel if and only if $|\xi_1\eta_2 - \eta_1\xi_2| = 0$. We will prove this using the Schwarz inequality $\langle X_1|X_2\rangle\langle X_2|X_1\rangle \leq \|X_1\|^2 \cdot \|X_2\|^2$ as follows:

$$\begin{aligned}\langle X_1|X_2\rangle\langle X_2|X_1\rangle &= (\xi_1\bar{\eta}_1 + \xi_2\bar{\eta}_2)(\bar{\xi}_1\eta_1 + \bar{\xi}_2\eta_2) \\ &= |\xi_1|^2|\eta_1|^2 + \xi_1\bar{\eta}_1\bar{\xi}_2\eta_2 + \xi_2\bar{\eta}_2\bar{\xi}_1\eta_1 + |\xi_2|^2|\eta_2|^2,\end{aligned}\quad (5)$$

where, i.e., $\bar{\xi}$ is the complex conjugate of ξ . The product of the norms of these vectors is given by

$$\|X_1\|^2 \cdot \|X_2\|^2 = |\xi_1|^2|\eta_1|^2 + |\xi_1|^2|\eta_2|^2 + |\xi_2|^2|\eta_1|^2 + |\xi_2|^2|\eta_2|^2. \quad (6)$$

If X and Y are parallel, then we have $\langle X_1|X_2\rangle\langle X_2|X_1\rangle = \|X_1\|^2 \cdot \|X_2\|^2$, which implies that

$$\xi_1\bar{\eta}_1\bar{\xi}_2\eta_2 + \xi_2\bar{\eta}_2\bar{\xi}_1\eta_1 = |\xi_1|^2|\eta_2|^2 + |\xi_2|^2|\eta_1|^2 \implies |\xi_1\eta_2 - \eta_1\xi_2|^2 = 0. \quad (7)$$

That is, X_1 and X_2 are parallel if, and only if,

$$\det \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} = 0, \quad (8)$$

where \det denotes the determinant. We note that the area of a parallelogram spanned by two vectors is equal to the value of their 2-by-2 determinant.

Now we set out to generalize this simple result to a larger bipartite product space. Let $X_1 = (\xi_1, \xi_2, \dots, \xi_{N_2})$ and $X_2 = (\eta_1, \eta_2, \dots, \eta_{N_2})$ be two vectors defined on the Hilbert space $\mathcal{H} = \mathbf{C}^{N_2}$. Again by using the Schwarz inequality, we get

$$\begin{aligned}\langle X_1|X_2\rangle\langle X_2|X_1\rangle &= (\xi_1\bar{\eta}_1 + \xi_2\bar{\eta}_2 + \dots + \xi_{N_2}\bar{\eta}_{N_2})(\bar{\xi}_1\eta_1 + \bar{\xi}_2\eta_2 + \dots + \bar{\xi}_{N_2}\eta_{N_2}) \\ &= |\xi_1|^2|\eta_1|^2 + \xi_1\bar{\eta}_1\bar{\xi}_2\eta_2 + \xi_1\bar{\eta}_1\bar{\xi}_3\eta_3 + \xi_2\bar{\eta}_2\bar{\xi}_1\eta_1 + |\xi_2|^2|\eta_2|^2 \\ &\quad + \dots + \xi_{N_2-1}\bar{\eta}_{N_2-1}\bar{\xi}_{N_2}\eta_{N_2} + \xi_{N_2}\bar{\eta}_{N_2}\bar{\xi}_{N_2-1}\eta_{N_2-1} \\ &\quad + |\xi_{N_2}|^2|\eta_{N_2}|^2,\end{aligned}\quad (9)$$

and, in the same way as we have done above, we calculate the product of the norms of these vector as follows:

$$\begin{aligned}\|X_1\|^2 \cdot \|X_2\|^2 &= |\xi_1|^2(|\eta_1|^2 + |\eta_2|^2 + \dots + |\eta_{N_2}|^2) \\ &\quad + |\xi_2|^2(|\eta_1|^2 + |\eta_2|^2 + \dots + |\eta_{N_2}|^2) \\ &\quad + \dots + |\xi_{N_2}|^2(|\eta_1|^2 + |\eta_2|^2 + \dots + |\eta_{N_2}|^2) \\ &= |\xi_1|^2|\eta_1|^2 + |\xi_1|^2|\eta_2|^2 + \dots + |\xi_1|^2|\eta_{N_2}|^2 \\ &\quad + |\xi_2|^2|\eta_1|^2 + |\xi_2|^2|\eta_2|^2 + \dots + |\xi_2|^2|\eta_{N_2}|^2 \\ &\quad + \dots + |\xi_{N_2}|^2|\eta_1|^2 + |\xi_{N_2}|^2|\eta_2|^2 + \dots + |\xi_{N_2}|^2|\eta_{N_2}|^2.\end{aligned}\quad (10)$$

Again, if X_1 and X_2 are parallel, then we have $\langle X_1|X_2\rangle\langle X_2|X_1\rangle = \|X_1\|^2 \cdot \|X_2\|^2$ which, after some simplification, can be rewritten as follows:

$$\begin{aligned}&|\xi_1|^2|\eta_2|^2 - \xi_1\eta_2\bar{\eta}_1\bar{\xi}_2 - \xi_2\eta_1\bar{\eta}_2\bar{\xi}_1 + |\xi_2|^2|\eta_1|^2 + \dots + |\xi_{N_2-1}|^2|\eta_{N_2}|^2 \\ &- \xi_{N_2-1}\eta_{N_2}\bar{\eta}_{N_2-1}\bar{\xi}_{N_2} - \xi_{N_2}\eta_{N_2-1}\bar{\eta}_{N_2}\bar{\xi}_{N_2-1} + |\xi_{N_2}|^2|\eta_{N_2-1}|^2 \\ &= |\xi_1^2\eta_2^2 - \xi_1\eta_2|^2 + \dots + |\xi_{N_2-1}\eta_{N_2} - \xi_{N_2}\eta_{N_2-1}|^2 = 0.\end{aligned}$$

That is, X_1 and X_2 are parallel if, and only if,

$$|\xi_1\eta_2 - \eta_1\xi_2| = |\xi_1\eta_3 - \eta_1\xi_3| = \dots = |\xi_{N_2-1}\eta_{N_2} - \eta_{N_2-1}\xi_{N_2}| = 0. \quad (11)$$

This result implies that

$$\det \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} = \dots = \det \begin{pmatrix} \xi_{N_2-1} & \xi_{N_2} \\ \eta_{N_2-1} & \eta_{N_2} \end{pmatrix} = 0$$

if the vectors are parallel.

To establish a relation between the Schwarz inequality and the concurrence, let us consider the quantum system $\mathcal{Q}_2^p(N_1, N_2)$ be a pure, bipartite quantum system. Then, the concurrence can be written as [17]

$$\mathcal{C}(\mathcal{Q}_2^p(N_1, N_2)) = \left(\mathcal{N} \sum_{l_1 > k_1}^{N_1} \sum_{k_1=1}^{N_1} \sum_{l_2 > k_2}^{N_2} \sum_{k_2=1}^{N_2} \left| T \begin{pmatrix} k_1 & l_1 \\ k_2 & l_2 \end{pmatrix} \right|^2 \right)^{\frac{1}{2}} \quad (12)$$

where $T \begin{pmatrix} k_1 & l_1 \\ k_2 & l_2 \end{pmatrix} = \det \begin{pmatrix} \alpha_{k_1, k_2} & \alpha_{k_1, l_2} \\ \alpha_{l_1, k_2} & \alpha_{l_1, l_2} \end{pmatrix}$ is a second order minor of the $2 \times N_2$ matrix

$$\begin{pmatrix} \alpha_{k_1, 1} & \alpha_{k_1, 2} & \dots & \alpha_{k_1, N_2-1} & \alpha_{k_1, N_2} \\ \alpha_{l_1, 1} & \alpha_{l_1, 2} & \dots & \alpha_{l_1, N_2-1} & \alpha_{l_1, N_2} \end{pmatrix}, \quad (13)$$

where \mathcal{N} is a normalization constant. We recognize the expression (12) as the sum of all parallelograms computed above. Hence, the concurrence is zero only if the Schwartz inequality is satisfied with equality for all pairs of vectors $X_{k_1} = (\alpha_{k_1, 1}, \alpha_{k_1, 2}, \dots, \alpha_{k_1, N_2})$ and $X_{l_1} = (\alpha_{l_1, 1}, \alpha_{l_1, 2}, \dots, \alpha_{l_1, N_2})$. This implies that all the vectors X_{k_1} and X_{l_1} are parallel. If so, the state is obviously separable because this means that the state can be written

$$|\Psi\rangle = (\alpha_{1,1}|1_1\rangle + \dots + \alpha_{N_1,1}|N_{11}\rangle) \otimes (|1_2\rangle + \alpha_{1,2}/\alpha_{1,1}|2_2\rangle + \dots + \alpha_{1,N_2}/\alpha_{1,1}|N_{22}\rangle). \quad (14)$$

We also see that the concurrence for a bipartite, pure state, loosely speaking, has the geometrical interpretation of the summed pairwise deviation from parallelism of all the vectors X_{k_1} and X_{l_1} .

4 The Schwarz inequality and concurrence of a general pure three-partite state

Let us now see what happens if we consider the simplest example of a three-partite system. The simplest tripartite system, consisting of three qubits, is denoted $\mathcal{Q}_3^p(2, 2, 2)$. The concurrence of this state is then given by [17]

$$\mathcal{C}(\mathcal{Q}_3^p(2, 2, 2)) = \left(\mathcal{N} \sum_{j=1}^3 \sum_{l > k}^{2,2} \sum_{k=1,1}^{2,2} \sum_{l_j > k_j}^2 \sum_{k_j=1}^2 \left| T \begin{pmatrix} k_j & l_j \\ k \neq k_j & l \neq l_j \end{pmatrix} \right|^2 \right)^{\frac{1}{2}} \quad (15)$$

Where $T \begin{pmatrix} k_1 & l_1 \\ k \neq k_1 & l \neq l_1 \end{pmatrix}$ is a minor of the 2×4 matrix

$$\begin{pmatrix} \alpha_{k_1, 1, 1} & \alpha_{k_1, 1, 2} & \alpha_{k_1, 2, 1} & \alpha_{k_1, 2, 2} \\ \alpha_{l_1, 1, 1} & \alpha_{l_1, 1, 2} & \alpha_{l_1, 2, 1} & \alpha_{l_1, 2, 2} \end{pmatrix}, \quad (16)$$

and where the two-digit indices k and l run from 1,1 to 2,2, and, where of course the only possibility is that $k_1 = 1$ and $l_1 = 2$. In the same manner, $T \begin{pmatrix} k_3 & l_3 \\ k \neq k_3 & l \neq l_3 \end{pmatrix}$ is a minor of the matrix

$$\begin{pmatrix} \alpha_{1,1,k_3} & \alpha_{1,2,k_3} & \alpha_{2,1,k_3} & \alpha_{2,2,k_3} \\ \alpha_{1,1,l_3} & \alpha_{1,2,l_3} & \alpha_{2,1,l_3} & \alpha_{2,2,l_3} \end{pmatrix}, \quad (17)$$

etc.

If we apply the Schwarz inequality to all the combinations of the above pairs of vectors, then we get the desired result. The interpretation of the result is that $T \begin{pmatrix} k_j & l_j \\ k \neq k_j & l \neq l_j \end{pmatrix}$ generates the minor determinants that establish whether system \mathcal{Q}_j^p is separable from the rest of the system. Again we can easily generalize the result above to a general, pure, three-partite state $\mathcal{Q}_3^p(N_1, N_2, N_3)$. Then concurrence of this state is given by [17]

$$\mathcal{C}(\mathcal{Q}_3^p(N_1, N_2, N_3)) = \left(\mathcal{N} \sum_{j=1}^3 \sum_{l>k} \sum_k \sum_{l_j>k_j}^{N_j} \sum_{k_j=1}^{N_j} |T \begin{pmatrix} k_j & l_j \\ k \neq k_j & l \neq l_j \end{pmatrix}|^2 \right)^{\frac{1}{2}} \quad (18)$$

where, e.g., the indices k and l for $j = 1$ run through the $N_2 N_3$, two-digit numbers 1,1 to N_2, N_3 .

From the above discussion we can see that the equality in Schwarz inequality can be used as a criterion for separability, and the deviation from equality, in the sense outlined above, can be used as measure of entanglement which coincides with generalized concurrence and our entanglement tensor for bi- and three-partite states.

5 Conclusion

We have discussed the relation between Schwarz inequality (or, rather, equality) and concurrence for bi- and three-partite states and possible generalization to multi-partite states. This relation helps us visualize the geometrical properties of concurrence and the relation is directly related to the geometry of the Hilbert space as a normed complex space with an inner-product defined on it. Moreover, we have shown that the deviation from the Schwarz inequality upper bound (perhaps this bound can be called the ‘‘Schwarz equality’’) can be used as measure of entanglement for concurrence for bi- and three-partite states.

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